The King's School Canterbury

13+ Scholarship Examinations 2017

MATHEMATICS II

1 hour (including five minutes suggested reading time)

Use the reading time wisely; gain an overview of the paper and start to think of how you will answer the questions.
Do as many questions as you can (clearly numbered) on the lined paper provided. Clearly name each sheet used.
*The questions are not of equal length or mark allocation. Move on quickly if stuck. Y**ou are not expected to finish everything.***
*You are expected to use a calculator where appropriate, but you must show **full and clear working**, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*
Complete solutions are preferable to fragments.

This paper has seven questions.

1 Every pupil at King's is either a scholar or not, and (for this question) is either a girl or a boy.

Girl scholars always tell the truth and girl non-scholars always lie.
Boys are the other way around i.e. scholars always lie, non-scholars always tell the truth.

The Girl says: I am a scholar.

The Boy says: that is true.

What are they each? Is there more than one answer? Show your reasoning carefully.

2 Baldrick has twice as many turnips as he has carrots.

He eats ten pieces of each kind, and now has three times as many turnips as carrots.

How many turnips did he originally have?

3 Boris used to sell spinach to the UK and Europe **only**, with 99% of it going to Europe.

Even post-Brexit he still sells 95% to Europe, while selling exactly the same amount in the UK as before; he does not sell any to anyone else.

Explain carefully why this is a worrying development, showing any calculations.

4 Suppose that *x* and *y* are non-zero numbers such that

$$\frac{5x+y}{x-5y}=3$$

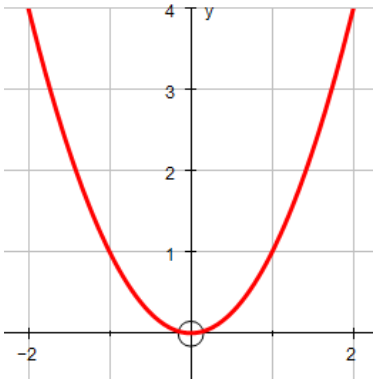
What is the value of

$$\frac{x+5y}{5x-y}$$

?

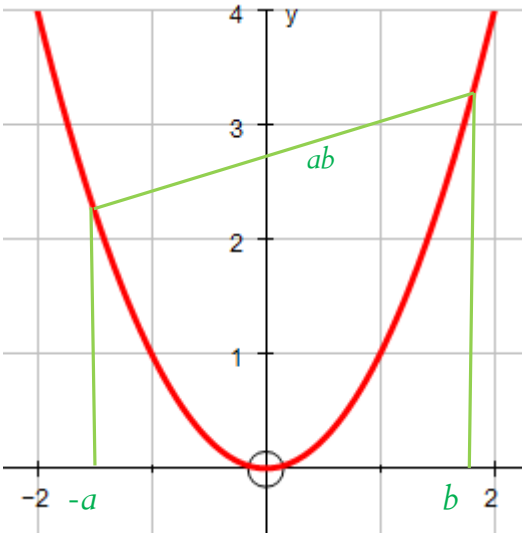
5 Professor Möbius draws a graph of $y = x^2$

(here shown from x values from -2 to 2).



Möbius claims that he can use this graph as a new way of multiplying two numbers together.

He says: "If I want to multiply a and b, I find the points on the graph at $x = -a$ and $x = b$. I then join these with a straight line, and where this crosses the y-axis will be the answer, ab."



Show carefully using algebra that Professor Möbius is correct. (Copy the diagram into your script as needed.)

[In this question you might find the following algebraic fact (the "difference of two squares") useful:

$$a^2 - b^2 = (a + b)(a - b)$$

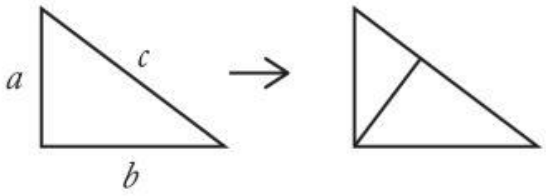
.]

6 In this question we are going to prove the Pythagorean Theorem, namely

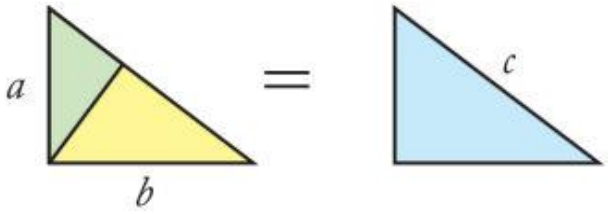
$$a^2 + b^2 = c^2$$

for a right-angled triangle of sides a, b, c (c is the **hypotenuse**). The proof you might reconstruct below is sometimes attributed to a twelve-year-old Albert Einstein.

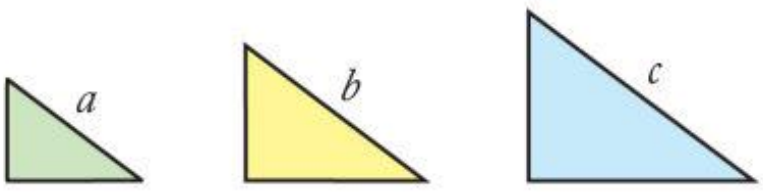
Draw a perpendicular line from the hypotenuse to the right angle, which divides the original right-angled triangle into two smaller right-angled triangles.



(a) What is the connection between the little triangle area, the area of the medium triangle and that of the original triangle?

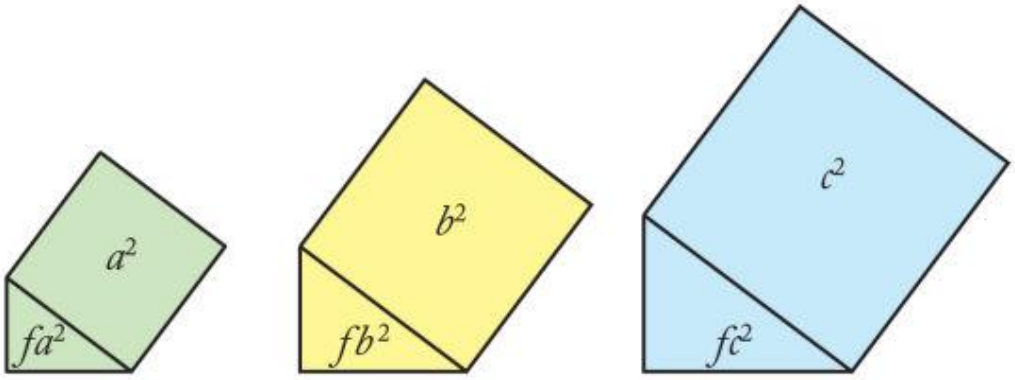


(b) Explain why the three triangles are in fact **similar** (same shape, not necessarily same size).



(c) Explain the following: because the triangles are similar, each will occupy the same fraction f of the area of the square on its hypotenuse.

(Restated symbolically, this observation says that the triangles have areas fa^2, fb^2, and fc^2, as indicated in the diagram.)



(d) Explain why $fa^2 + fb^2 = fc^2$.

(e) How does this give us the Pythagorean theorem?

7 On the pages overleaf is a type of poem called a **Sestina** written in a mediaeval language (Old Occitan) by Arnaut Daniel in about 1200.

Do not try to read the poem; we are only interested in the pattern of the last words on each line.

The main part of the poem consists of six stanzas (verses) where the last words on each line cycle according to a fixed rule. (For now, we ignore the "envoi" (the last three lines)).

The numbering (summarised again below) shows how the end-words cycle around from stanza to stanza.

I	II	III	IV	V	VI
1	6	3	5	4	2
2	1	6	3	5	4
3	5	4	2	1	6
4	2	1	6	3	5
5	4	2	1	6	3
6	3	5	4	2	1

(a) If the rearrangement continued, what would be the ordering for the seventh stanza?

(b) What might be the simplest rule that left nothing unchanged each time, but then returned to the starting arrangement after six stanzas?

(c) What happens if I try writing a sestina using the following rule for rearrangement: 123456 in the first stanza moves to 312645 in the second (and then repeating)?

Use the following information in the next two parts.

There are 720 = 6! possible rearrangements of six words.
There are 265 of these possible rearrangements where nothing ends up in the same position.
Of these, 120 will take six repeats to return to the starting point.
90 will take four repeats to return to the starting point.
40 will take three repeats to return to the starting point.
15 will take two repeats to return to the starting point.

(d) Suppose Arnaut Daniel chooses the rearrangement rule at random. What is the probability that he chooses a rearrangement where at least one word ends up in the same position as before?

(e) Another writer chooses at random and she picks one where nothing ends up in the same position. What is the probability that it will return to the starting point in less than six repeats?

Another way of generating last words in a poem follows this rule (with n taking number values 1 to 6)

Input n
Output:

- If n is even, write $\frac{n}{2}$ in that position
- If n is odd, write $(6 - \frac{n-1}{2})$ in that position.

(f) Starting with a first stanza 123456 as before write down a sequence of numbers to show how the next five stanzas' lines will end using this rule. Comment on your answers.

Lo ferm voler q'el cor m'**intra**	**1**	Stanza I
no'm pot ies becs escoissendre ni **ongla**	**2**	
de lausengier, qui pert per mal dir s'**arma**	**3**	
e car non l'aus batr'ab ram ni ab **verga**	**4**	
si vals a frau lai o non aurai **oncle**	**5**	
jauzirai joi, en vergier o dinz **cambra**	**6**	
Qan mi soven de la **cambra**	**6**	Stanza II
on a mon dan sai que nuills hom non **intra**	**1**	
anz me son tuich plus que fraire ni **oncle**	**5**	
non ai membre no'm fremisca, neis l'**ongla**	**2**	
aissi cum fai l'enfas denant la **verga**	**4**	
tal paor ai no'l sia trop de l'**arma**	**3**	
Del core li fos non de l'**arma**	**3**	Stanza III
e cossentis m'a celat deniz sa **cambra**	**6**	
que plus mi nafra'l cor que colps de **verga**	**4**	
car lo sieus sers lai on ill es non **intra**	**1**	
totz temps serai ab lieis cum carns et **ongla**	**2**	
e non creirai chastic d'amic ni d'**oncle**	**5**	
Anc la seror de mon **oncle**	**5**	Stanza IV
non amei plus ni tant per aqest'**arma**	**3**	
c'aitant vezis cum es lo detz de l'**ongla**	**2**	
s'a liei plagues volgr'esser de sa **cambra**	**6**	
de mi pot far l'amors q'inz el cor m'**intra**	**1**	
mieills a son vol c'om fortz de frevol **verga**	**4**	
Pois flori la seca **verga**	**4**	Stanza V
Ni d'en Adam mogron nebot ni **oncle**	**5**	
tant fin'amors cum cella q'el cor m'**intra**	**1**	
non cuig fos anc en cors ni eis en **arma**	**3**	
on q'ill estei fors on plaz' o dins **cambra**	**6**	
mos cors no' is part de lieis tant cum ten l'**ongla**	**2**	
C'aissi s'enpren e s'en**ongla**	**2**	Stanza VI
mos cors e lei cum l'escorss'en la **verga**	**4**	
q'ill m'es de joi tors e palaitz e **cambra**	**6**	
e non am tant fraire paren ni **oncle**	**5**	
q'en paradis n'aura doble joi m'**arma**	**3**	
si ja nuills hom per ben amar lai **intra**	**1**	
Arnautz tramet sa chanson d'**ongl**'e d'**oncle**	**2,5**	Envoi
a grat de lieis que de sa **verg'a** l'**arma**	**4,3**	(ignore this part)
son Desirat cuit pretz en **cambra intra**	**6,1**	

END OF PAPER

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The questions are not of equal length or mark allocation. Make sure you avoid spending too much time on any one question; don't get bogged down! Move on quickly if you get stuck. The paper is long; ***you are not expected to finish everything.***

Some of the later questions are more difficult, but not necessarily longer. Some questions are designed to test your ability to work with unfamiliar ideas, or familiar ones with a twist. Don't give up!

You are expected to use a calculator where appropriate, but you must show ***full and clear working****, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments. You can sometimes, however, manage to complete later parts of questions, even if you have failed to answer the earlier sections.

This paper has nine questions.

1 Vic is awarded £X prize money for scholarship performance and Bob is awarded £Y.

Vic has won more than Bob and they both win an odd number of pounds.

Find an algebraic expression for the smallest (whole) number of pounds Vic needs to give Bob so that Bob has more money than Vic.

2 There are several choirs singing at King's. Membership of choirs is linked as follows:

- All members of Crypt Choir are in Chapel Choir
- Almost all Madrigalia are in Crypt Choir
- All of Chapel Choir are in Nave Choir
- All of Becket singers are in Nave Choir.
- All Nave Choir and all Girls' Choir are in King's Chorus.
- Other groups include King's Men, King's Swingers and King's A Capella.

State whether each of the following is true or false (or you cannot determine an answer), giving brief explanations for each.

(a) I am in Crypt Choir, therefore I am in King's Chorus.

(b) I can be a Becket Singer and in Chapel Choir.

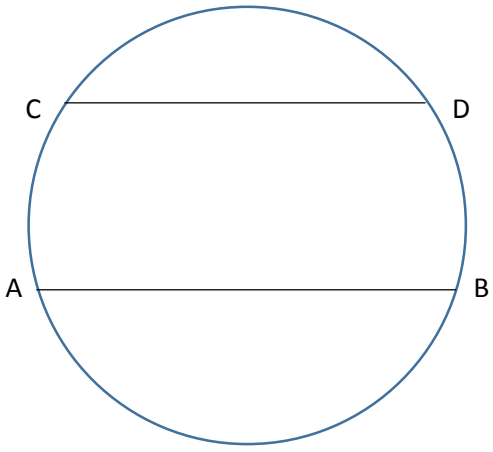
(c) I am in Crypt Choir, therefore I am in Madrigalia.

(d) I am not in the Nave Choir, therefore I am in neither Chapel nor Crypt.

(e) The Nave Choir is larger than the King's Chorus.

3 AB and CD are parallel chords (lines) on a circle of length 14cm and 10cm respectively. The chords are 6cm apart. Find the distance of the centre of the circle from the line AB.

NOT TO SCALE



[Hint: you may wish to draw a diagram and add some radii to it.]

4 Suppose that *W, X, Y, Z* are connected by the formula

$$W = \frac{XY}{Z}$$

If *X* is increased by 5%, *Y* is decreased by 68% and *Z* is decreased by 46%, work out the percentage change in *W*, stating whether it is an increase or a decrease.

5 Another school brings in a new scholarship selection method. For each subject for which they are entered, candidates pick (at random) a ball out of a bag containing 100 balls (numbered 1,2,3,…,100), and then replace it.

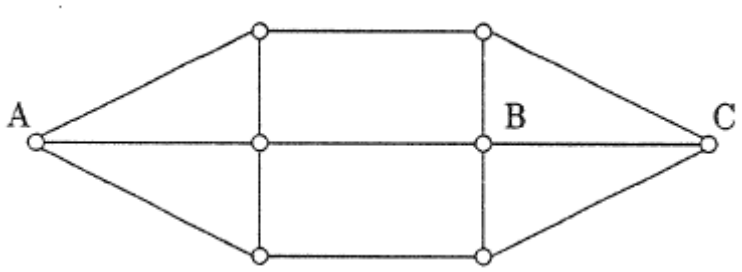
The candidate is classified as **weird** if he/she picks out ball 1 or ball 100 for any subject. Candidates are either weird or not weird.

So, we expect 2% of candidates entering only one subject to be weird.

Suppose the school has a very large number of scholarship candidates.

(a) Explain, showing working, why we expect 3.96% of candidates entering two subjects to be weird.
(b) Work out the expected percentage of weird candidates sitting three subjects.
(c) By extending any pattern in the calculations you have been doing, show that if candidates are put in for fifteen subjects then more than a quarter of them will be expected to be weird.
(d) What do you think is the pattern to these percentages as we increase the number of subjects taken. Why do you think this is so?

6 Some rats have a network of tunnels as shown. Consider a rat. Each night he sleeps at one of the junctions. Each day he moves to a neighbouring junction but chooses a path at random (each of equal likelihood from those available).



A rat starts at A.

(a) What is the probability that two nights later he is at B?

Another rat starts at B

(b) Calculate the probability that two nights later he is back at B again.

Two more rats start, one at C and the other at A

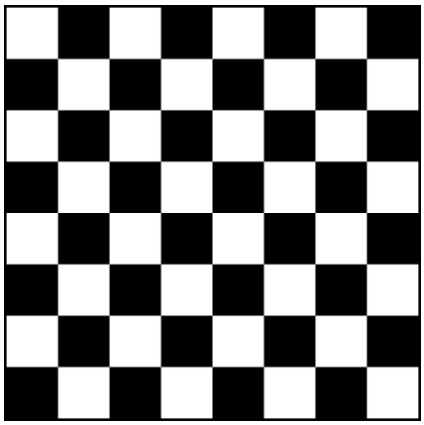
(c) Calculate the probability that two nights later they are at the same junction.

7 Alfred tosses a coin until he throws a head and then stops.

He claims that the probability that he stops on an even-numbered turn is ½ .

Explain why he is wrong and work out the actual probability that he stops on an even-numbered turn.

8 Consider a standard 8x8 chessboard.



(a) How many different ways are there of placing two identical rooks (pieces) on an 8×8 chessboard?

(b) Now suppose we have a white King and a black King, which are **not** allowed to occupy adjacent squares. In how many different ways can we place them on the chessboard?

NB the chessboard is of fixed orientation i.e. if two possible configurations of pieces differ by a rotation then we do count them as being different in both parts of this question.

9 A, B, C are three scholarship candidates, sitting the examinations, not necessarily each taking the same number of test papers.

A has a mean average score of 40 points.
B has a mean score of 50 points.
Taken together A and B have a mean score of 43 points.
Taken together A and C have a mean score of 44 points.
What is the greatest possible integer (whole number) mean score for B and C combined?

END OF PAPER