13+ Scholarship Examinations 2017

MATHEMATICS I

Seventy-five minutes (plus five minutes reading time)

Use the reading time wisely; gain an overview of the paper and start to think of how you will answer the questions.

Do as many questions as you can (clearly numbered) on the lined paper provided. Clearly name each sheet used.

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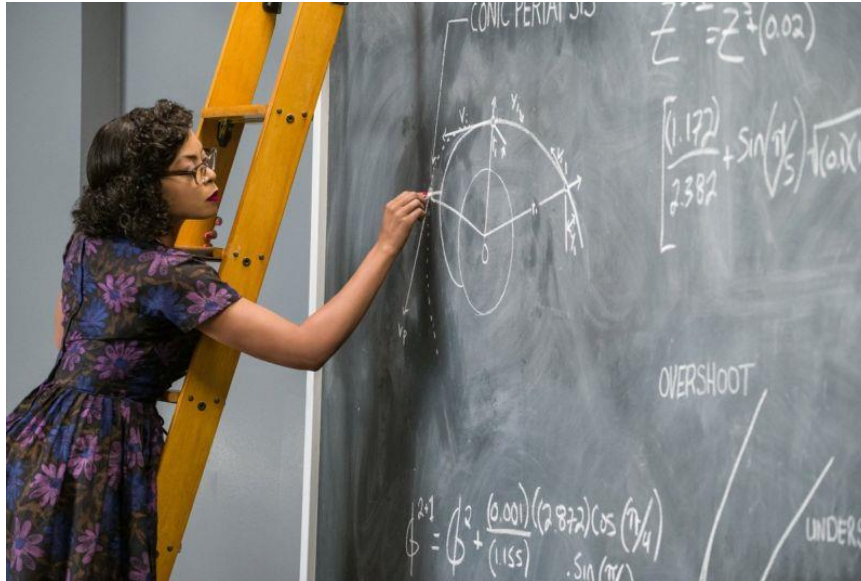
This paper has fourteen questions.

1   In the 2017 film *Hidden Figures*, NASA worker Katherine Johnson makes some calculations in order to find the flight path for the first American manned orbit of Earth.

In her calculations, there is a number used for the Earth's rotation: 0.25068 degrees per minute

Use this to calculate the length of a day in hours, i.e. the time taken for Earth to make one complete rotation.

When giving your answer, write down all the figures on your calculator.

2   (a)   Out of 50 scholarship candidates, 30 speak Gaelic, 20 speak Welsh, and 5 speak neither Gaelic nor Welsh. What is the ratio of those who speak both Gaelic and Welsh to those who speak neither?

         (b)   Out of 70 different scholarship candidates, 40 speak Gaelic, 30 speak Welsh, and x speak neither Gaelic nor Welsh. What is the ratio of those who speak both Gaelic and Welsh to those who speak neither? Explain your answer.

3   In a Senate (parliament) there are 100 senators. Each is either honest or crooked; some of the Senate is honest, and some of it is crooked. However, if you choose any two senators at random, you can be sure that at least one is crooked. How many honest senators are there?
Explain your reasoning.

4   Solve the following for x:

(a)
$$\frac{5x}{8} + \frac{1}{6} = \frac{109}{24}$$

(b)
$$x + 23 + 7(5 + 2x) = 9x - \frac{8}{5}(7x - 4)$$

5   This question is about a technique which first appeared in print in Luca Pacioli's *Summa de arithmetica*, a mathematics textbook published in 1494.

In money problems involving **compound** interest the **Rule of 72** gives us a quick way to calculate how long it will take for a quantity to double in size.

The rule states that a quantity growing at R % per year will double in size in roughly

$$t = \frac{72}{R}$$ years.

The King's School wishes to build an ivory tower for the scholars, so it borrows £1 million at an annual interest rate of 9%. For a long time, the school does not pay anything back, so the amount owed will grow.

(a)   Using the Rule of 72, how long does it suggest it will take for the debt to double?

(b)   An alternative to the Rule of 72 is

$$t = \frac{72 + \frac{R - 8}{3}}{R}$$

Using this rule instead, how long does it suggest it will take for the debt to double?

(c)   By choosing a suitable "percentage multiplier" m and trying out a couple of values for t, use the formula below to find another answer to the question: when is the amount doubled?

$$2000000 = 1000000 \times m^t$$

Compare your answer with those in (a) and (b).

6   Ludwig offers individual scholarship training courses.

Each one costs him £287 per person to run.

He wants to sell them to potential candidates with a discount offer of 30% off the published price, while still making a profit of 20% on his original cost.

What should the published price be?

7   Thirty candidates sit another scholarship examination (not at King's) which has three components: reading, writing and arithmetic.

Each candidate must sit at least one examination, but may take more.

- 11 sit the reading examination
- 14 sit writing
- 19 sit arithmetic
- 8 sit at least two examinations

How many sit all three?

8   (a) Solve the simultaneous equations:

$$25x - 4y = 100$$

$$16x + 9y = 144$$

(b) Hence solve the simultaneous equations below, giving **all** values for p and q:

$$\frac{p^2}{4} - \frac{q^2}{25} = 1$$

$$\frac{p^2}{9} + \frac{q^2}{16} = 1$$

9   I meet three scholarship candidates, Algy, Biggles and Carstairs; each of them either lies all the time or tells the truth all the time.

I know there is at least one of each and that only one of the three has been awarded a scholarship.

Algy says   Biggles did not win a scholarship
Biggles says   I did not win a scholarship
Carstairs says   I did win a scholarship

Who did in fact win? Show your reasoning carefully.

10   Below are some facts about the new UK £1 coin being issued from March 2017, which replaces the round one in use since 1983.

- It has twelve (slightly curved, not straight) edges.
- 1.5 billion of these coins will be issued by October
- Its thickness is 2.8mm
- It has variable width, between 23.03mm and 23.43mm
- Its mass or weight is 8.75g

Use whichever of these facts you need in the questions which follow.

(a)   What is the percentage increase in width from the minimum to the maximum?

(b)   Explain carefully why $1 \text{ m}^3 = 1000000000 \text{ mm}^3$.

(c)   By modelling the coin face as a regular dodecagon, use the formula below to find an approximate **volume** for one one-pound coin.

Area A of regular dodecagon (12-sided shape): $$A = 12(2 - \sqrt{3})r^2$$

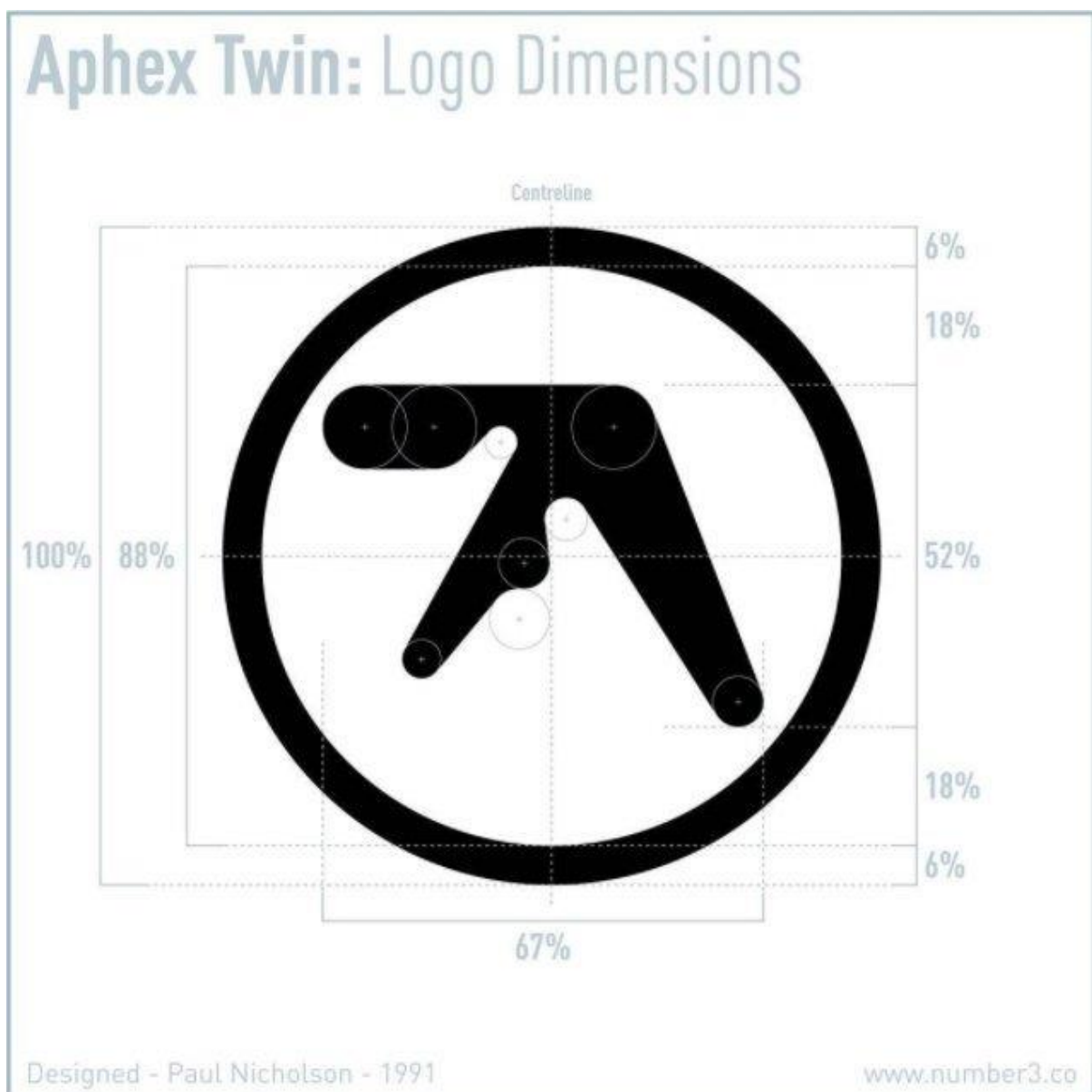where r is the "radius" of the coin (find this using a width value you choose).

(d)   Using your answers to parts (b) and (c), what is the **density** of one of these coins? [Give your answer in **kg/m³**.]

(e)   Stating any assumptions clearly, work out approximately how much space I would need to store the 1.5 billion new pound coins. Try and give some sense of how big your answer is, e.g. a classroom, a cathedral etc.

11 The electronic musician Aphex Twin has a logo (below); its designer recently revealed the details of its creation.

The **whole logo** is a black circle and a white interior with a black design at the centre. The central black part of the design takes up 17 % of the whole logo area.

What is the total black area as a percentage of the whole logo in this design? Show all your working clearly, using relevant data in the diagram below.

Aphex Twin: Logo Dimensions

Centreline

6%
18%
52%
18%
6%

100% 88%

67%

Designed - Paul Nicholson - 1991 www.number3.co

12   Carruthers is going to sit the scholarship examinations and buys some new ink for his fountain pen. The new ink lasts 50% longer (imagine it is measured in units of, say, metres per millilitre (m/ml)). The new ink is, however, 20% more expensive than the old ink. Carruthers uses the new ink to write all his examinations; by what percentage does he save money by not using the old ink?

13   [In this question you might like to use the notation $n!$ representing "n factorial", namely

$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \quad = \quad n! \qquad]$$

(a) Explain why there are 96 ways of rearranging the repeated letters A, C, I, D, S in the word

ARCHIDIDASCALUS

so that the new word still looks the same as the original.

(b)   Hence show that there are 13621608000 essentially different rearrangements of the word

ARCHIDIDASCALUS

(c)   Estimate, showing working, how long it would take to write down all the different rearrangements.

14   In another school some boys and girls sit some optional scholarship papers in mathematics and English, and these are graded simply Pass or Fail. The teachers collate the results in a table.

	Math		Eng			Pass Rate %		
	pass	fail	pass	fail		Math	Eng	Overall
Boys	16	4	1	4		80	?	?
Girls	9	1	4	6		90	?	?

(a)   The pass rates for mathematics have already been worked out. Write down (not on the question paper) the pass rates for English. Make a comment on the mathematics and English performance of the boys and girls.

(b)   Now work out the combined pass rates i.e. one figure for boys and one for girls. What do you notice about these new figures?

Then the teachers realise they have missed out some results, and add them to the table.

	Math		Eng			Pass Rate %		
	pass	fail	pass	fail		Math	Eng	Overall
Boys	16	4	1	4		80	?	?
Girls	9	1	8	7		90	?	?

(c)   Work out any changed pass rates for this table.

(d)   Write some comments on the pass rates.

END OF PAPER

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Do as many questions as you can (clearly numbered) on the lined paper provided. Clearly name each sheet used. You are encouraged to attempt these questions in order.

The questions are not of equal length or mark allocation. Make sure you avoid spending too much time on any one question; don't get bogged down. Move on quickly if you get stuck. The paper is quite long; you are not necessarily expected to finish everything.

Some of the later questions are more difficult, but not necessarily longer. Some questions are designed to test your ability to work with unfamiliar ideas, or familiar ones with a twist. Don't give up.

*You are expected to use a calculator where appropriate, but you must show **full and clear working**, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers: merely writing down an answer might score very few marks.*

Complete solutions are preferable to fragments. You can sometimes, however, manage to complete later parts of questions, even if you have failed to answer the earlier sections.

This paper has eleven questions.

1

Inigo Montoya is looking for a man with six fingers on his right hand.

He goes to a village and rounds up the men living there. They each raise their right hand, having either five or six fingers.

Inigo counts 89 heads and 448 fingers. How many men with six-fingered hands are there in the village?

2

In January 2016 the discovery of a new largest prime number was announced. It is equal to

$$2^{74,207,281} - 1$$

and, when written out in full has 22,338,618 digits. If you were to write down all of the digits of this new prime number, how long would it take you?

[Work out an estimate and give your final answer in appropriate units.]

3

Solve the following for x:

(a)
$$7(3x + 2) = 119$$

(b)
$$\frac{3x}{4} - \frac{1}{10} = \frac{1}{5}$$

(c)
$$4 - 3(1 - x) = 5x + \frac{1}{4}(3x + 26)$$

4 Suppose we define an operation $*$ as
$$a*b = a^3 b^4$$

(a) What is the value of $\dfrac{2*4}{4*2}$?

(b) What is the value of $\dfrac{2*(3*4)}{(4*3)*2}$?

5 (a) Solve the simultaneous equations:

$$5x + 4y = 7$$
$$5y - 2x = 17$$

(8) Hence solve the simultaneous equations below, giving **all** values for p and q .

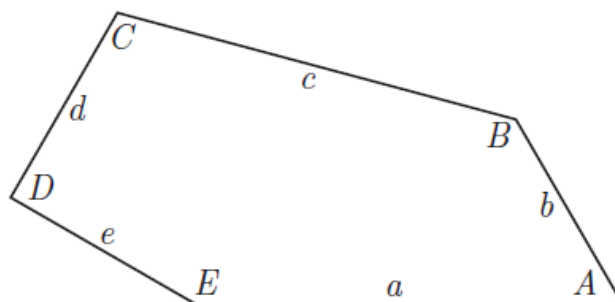$$\frac{5}{p} + 4q^2 = 7$$

$$5q^2 - \frac{2}{p} = 17$$

6

This Spring two rowers broke a world record at the King's School Recreation Centre, for the fastest one million metre row.

The pair beat the fastest one million metre row record by one hour, 19 minutes and 20 seconds with a time of three days 15 hours 34 minutes and 48 seconds.

If we assume the **previous** record holders rowed at a constant speed, how many metres did they have left to row when they reached this new record time?

7 In 2015 a paper was published introducing the fifteenth example of a pentagon which can be used to tile the plane (fourteen other ones were already known).

In the paper the authors find the five internal angles for this pentagon A, B, C, D, E by solving the following system of equations:

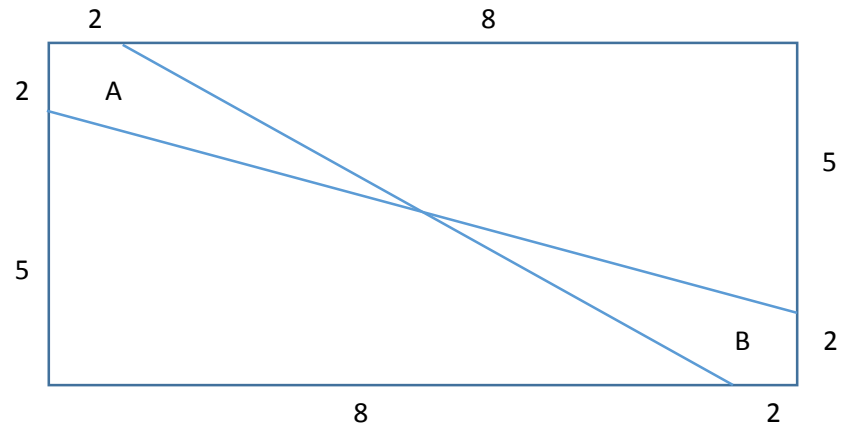2A + B + C = 360

2E + A = 360

2D + 180 = 360

2C + E = 360

2B + D = 360

Calculate each of the angles A, B, C, D, and E.

8

Calculate the total area A + B contained between the two straight lines inside the 10x7 rectangle below

NOT TO SCALE

9

[In this question you might like to use the notation $n!$ representing n factorial, namely

$$n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1 \quad = \quad n!$$]

(a) Consider the different rearrangements of the letters of the word

SCHOLAR

Suppose I write down **40%** of all the possible rearrangements of this word. Show that I have made 2016 different rearrangements.

(b) Now consider **all** the different rearrangements of the letters of the word

ANAGRAM

Explain carefully why there are 840 different rearrangements of this word.

10 [In this question, ignore any historical changes to the calendar and follow the calendar structure given in the question.]

2016 is a leap year (with one extra day, 29 February). Every year that is exactly divisible by four is a leap year, except for years that are exactly divisible by 100, but these centurial years are leap years if they are exactly divisible by 400. For example, the years 1700, 1800, and 1900 are not leap years, but the year 2000 is.

(a) (i) William died in 1616. Not including the year of his death, how many leap years have there been since, up to and including this year(2016)?

(ii)   In which year will the 2016th leap year be after this year (2016)?

(b) Use the information given above and (giving full working) show that the mean average number of days per year is 365.2425.

(c) Earth's rotation is gradually slowing down by around two thousandths of a second per day. Assuming this continues to be true, estimate how long it would be before the Earth stops spinning. Give your answer in appropriate units.

11

On an island there are two types of man, Sanders and Trumps.

Sanders always tell the truth; Trumps always lie.

(a) I meet a man and ask:

Are you a Trump or a Sanders?

His reply:

I'm not telling you! (and runs off)

What is he? Explain your answer.

(b) I meet another man and ask:

Do you ever answer no to questions?

He answered (yes or no) and then I knew for certain. What is he? Explain your answer.

END OF PAPER